

REMARKS ON GENERATORS OF ANALYTIC SEMIGROUPS[†]

BY

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ABSTRACT

This paper contains two new characterizations of generators of analytic semigroups of linear operators in a Banach space. These characterizations do not require use of complex numbers. One is used to give a new proof that strongly elliptic second order partial differential operators generate analytic semigroups in L^p , $1 < p < \infty$, while the sufficient condition in the other characterization is meaningful in the case of nonlinear operators.

§1. Introduction

Let X be a Banach space with norm $\| \cdot \|$, A be a densely defined linear operator in X and consider the initial value problem

$$(IVP) \quad \begin{cases} u' + Au = 0, \\ u(0) = x. \end{cases}$$

We recall some classical results concerning this initial value problem in a form which is convenient for what follows.

We denote by $\kappa(A)$ the set of all scalars λ for which $I + \lambda A$ has a bounded everywhere defined inverse, where I denotes the identity mapping on X . At this point we do not specify the scalar field; it may be either the real or the complex field. For $\lambda \in \kappa(A)$ we set

$$(1) \quad J_\lambda = (I + \lambda A)^{-1}.$$

The Hille–Yosida–Phillips theory implies that if $\kappa(A) \supset (0, \lambda_0)$ for some $\lambda_0 > 0$ and either of the equivalent conditions

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$$(2) \quad \begin{aligned} &\exists M > 0, \omega \in \mathbf{R} \text{ such that } \|J_\lambda^n\| \leq M(1 - \lambda\omega)^{-n} \\ &\text{for } 0 < \lambda < \lambda_0 \text{ and } n = 1, 2, \dots, \end{aligned}$$

or

$$(2)' \quad \exists M > 0 \text{ such that } \|J_\lambda^n\| \leq M \text{ for } 0 < \lambda < \lambda_0, n\lambda \leq 1 \text{ and } n = 1, 2, \dots$$

holds, then for each $x \in X$

$$(3) \quad S(t)x = \lim_{n \rightarrow \infty} J_{t/n}^n x$$

exists uniformly for t in bounded subsets of $[0, \infty)$. (By convention J_0 will be the identity on X rather than the identity on the domain $D(A)$ of A .) Moreover, if $x \in D(A)$, $u(t) = S(t)x$ is a classical solution of (IVP) on $[0, \infty)$, and S is a strongly continuous semigroup on X . Finally $-A$ is the infinitesimal generator of S and (2) or (2)' are equivalent to $-A$ being an infinitesimal generator of a strongly continuous semigroup S .

We are interested in the possibility of differentiating $S(t)x$ for $x \in X$. Everywhere in this section we assume $\kappa(A) \supset (0, \lambda_0)$ and either (2) or (2)' and so we have also (3). The strong condition $t \rightarrow S(t)x \in C^1([0, \infty); X)$ for all $x \in X$ is equivalent to $D(A) = X$ which is of no interest here. The weaker condition that $t \rightarrow S(t)x \in C^1((0, \infty); X)$ for each $x \in X$ may be restated as either $t \rightarrow S(t)x \in C^\infty((0, \infty); X)$ for all $x \in X$ or $S(t)X \subset D(A)$ for $t > 0$. Semigroups with these last properties are called C^∞ semigroups and their infinitesimal generators are characterized in [9]. The general class of C^∞ semigroups has not played a strong role in the applications of the theory so far. However, a more restricted class has enjoyed a wide range of applications, most notably to semilinear and quasilinear parabolic problems (see e.g. [7], [12], [6]). These are the *holomorphic* or *analytic* semigroups, and are the main interest of this note.

There is a variety of ways to define the concept of an analytic semigroup. If the underlying space is complex, it suffices to require that $t \rightarrow S(t)$ extends holomorphically from $t > 0$ to the sector $\{t \neq 0 : |\arg t| < \theta\}$ of the complex plane for some $\theta > 0$ and that for every $x \in X$, $S(z)x \rightarrow x$ as $z \rightarrow 0$ in this sector. If X is a real Banach space, one can complexify and ask that the complexified S be holomorphic in the above sense. This last idea seems unnatural to us when the application one has in mind is to, say, a semilinear problem $u' + Au + f(u) = 0$, $u(0) = x$, where f does not have a natural complexification.

A way to characterize analytic semigroups which ignores the scalar field in use is to require

(4) There is a dense subset D of X and $C > 0$ such that

$$\|AS(t)x\| \leq Ct^{-1}\|x\| \quad \text{for } 0 < t < 1, \quad x \in D.$$

Since $AS(t)$ is closed (4) implies that $AS(t)$ is everywhere defined for $0 < t$ and $\|AS(t)\| \leq Ct^{-1}$ for $0 < t < 1$.

In so far as we know, the published characterizations of the infinitesimal generators of analytic semigroups are all in terms of the complex resolvent of A . For example, in terms of $\kappa(A)$ and J_λ , S is analytic if and only if

$$(5) \quad \left\{ \begin{array}{l} \exists r, M > 0 \text{ such that } \kappa(A) \supset \Omega_r = \{\lambda : \lambda \in \mathbb{C}, \operatorname{Re} \lambda > 0 \text{ and } |\lambda| < r\} \\ \text{and } \|J_\lambda\| \leq M \text{ for } \lambda \in \Omega_r. \end{array} \right.$$

Here X has been complexified if it was originally a real Banach space. The characterization (5) is easily seen to be equivalent to the usual characterization in terms of the resolvent of $-A$ (see e.g. [10], [14]) via the relation

$$J_\lambda = (I + \lambda A)^{-1} = \lambda^{-1}(\lambda^{-1} + A)^{-1}.$$

Some observations concerning verification of (2) and (2)' in applications are in order. First (2)' has been stated only for later convenience. Secondly, if there is an equivalent norm $\|\cdot\|$ for which $\|J_\lambda\| \leq (1 - \lambda\omega)^{-1}$ for $0 < \lambda < \lambda_0$, then (2) holds. This is what is checked in practice. The only cases we know of which (2) has been established without exhibiting an equivalent norm for which $M = 1$ works, are those in which (5) is what is checked. Complex analytic arguments can be used to verify (2) when (5) holds, or else one shows via complex analytic arguments that if (5) holds, $-A$ generates a strongly continuous semigroup S and then (2) follows from the necessity of the Hille–Yosida–Phillips conditions.

We present here some simple characterizations of the infinitesimal generators of semigroups satisfying (4) which do not refer to complex numbers. These are Theorems 1 and 2 in the next section. Led by these considerations we have found a simple proof of the fact that second order scalar elliptic operators generate analytic semigroups in L^p spaces. This is presented in §3, where there are also further remarks concerning this problem.

Our original motivation in this inquiry was a desire to understand “regularizing effects” in known linear problems in such a way as to shed some new light on nonlinear problems. The contribution of this note in this direction is unfortunately rather small, but it may help to bring these concerns into focus.

§2. The abstract results

Our first characterization of the generator of an analytic semigroup is:

THEOREM 1. Let $-A$ be the infinitesimal generator of a strongly continuous semigroup $S(t)$ on X . Then $S(t)$ is an analytic semigroup if and only if for some C , $\lambda_0 > 0$

$$(2.1) \quad \|AJ_\lambda^n\| \leq \frac{C}{n\lambda} \quad \text{for } 0 < \lambda < \lambda_0, \quad n\lambda \leq 1 \quad \text{and} \quad n = 1, 2, \dots.$$

PROOF. Since $-A$ is the infinitesimal generator of S , (2)' and (3) hold. Assuming (2.1), if $x \in D(A)$ then

$$(2.2) \quad \|AJ_{t/n}^n x\| = \|J_{t/n}^n Ax\| \leq \frac{C}{t} \|x\| \quad \text{for } 0 < t < 1 \quad \text{and} \quad n \text{ large enough.}$$

Using (3) and the fact that A is closed we conclude that

$$\|AS(t)x\| = \|S(t)Ax\| \leq \frac{C}{t} \|x\| \quad \text{for } x \in D(A).$$

Thus (4) holds and S is analytic.

For the converse, if $S(t)$ is analytic there are constants $C_1 > 0$ and $\omega_1 \in \mathbf{R}$ such that

$$(2.3) \quad \|AS(t)\| \leq C_1 t^{-1} e^{\omega_1 t} \quad \text{for } t > 0.$$

This is a simple consequence of (4) and the growth condition $\|S(t)\| \leq Me^{\omega t}$ which follows from (2). Moreover, for any strongly continuous semigroup we have (see e.g. [10], [14])

$$(2.4) \quad J_\lambda = \lambda^{-1} \int_0^\infty e^{-t/\lambda} S(t) dt$$

where the integral converges in the strong operator topology if $\lambda > 0$ and $\lambda\omega < 1$. A simple induction establishes that

$$(2.5) \quad J_\lambda^{n+1} = \lambda^{-1} \int_0^\infty \left(\frac{t}{\lambda}\right)^n \frac{1}{n!} e^{-t/\lambda} S(t) dt.$$

Acting on both sides of (2.5) with A and using (2.3) we find

$$(2.6) \quad \|AJ_\lambda^{n+1}\| \leq \frac{C_1}{\lambda^2} \int_0^\infty \left(\frac{t}{\lambda}\right)^{n-1} \frac{1}{n!} e^{-(1/\lambda - \omega)t} dt = \frac{C_1}{n\lambda} \left(\frac{1}{1 - \lambda\omega}\right)^n$$

and (2.1) follows. ■

REMARKS. T. Kato informed us that he had previously observed the result of Theorem 1. Our formulation of it arose from consideration of nonlinear

problems. The condition (5) is not meaningful for nonlinear A , while (2.1) extends easily to this case. In fact one can verify the appropriate version of (2.1) if $A = \partial\varphi$ is the subdifferential of a lower semicontinuous convex functional on a Hilbert space (see Appendix 1). In other nonlinear problems this criterion appears to be too difficult to check (in a way analogous to (2) with $M > 1$).

The next result follows a different track. Given an operator A in X we form the operator \mathcal{A} in $X \times X$ corresponding to the 2×2 matrix

$$(2.7) \quad \mathcal{A} = \begin{pmatrix} A & 0 \\ A & A \end{pmatrix};$$

that is $\mathcal{A}(x, y) = (Ax, Ax + Ay)$ for $(x, y) \in D(A) \times D(A)$.

THEOREM 2. *Let A be a linear densely defined operator in X . Then $-A$ is the infinitesimal generator of an analytic semigroup on X if and only if $-\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup on $X \times X$.*

PROOF. We indicate two proofs. For the first one observe that $\kappa(A) = \kappa(\mathcal{A})$ and with $J_\lambda = (I + \lambda A)^{-1}$, $\mathcal{J}_\lambda = (I + \lambda \mathcal{A})^{-1}$ we have

$$(2.8) \quad \mathcal{J}_\lambda^n = \begin{pmatrix} J_\lambda^n & 0 \\ -n\lambda A J_\lambda^{n+1} & J_\lambda^n \end{pmatrix} \quad \text{for } \lambda \in \kappa(A), \quad n = 1, 2, \dots$$

Thus \mathcal{A} satisfies (2)' if and only if J_λ^n and $n\lambda A J_\lambda^{n+1}$ remain uniformly bounded for small $\lambda > 0$ and $n\lambda \leq 1$, $n = 1, 2, \dots$. Theorem 2 thus follows from Theorem 1.

The second proof arises from the idea behind the theorem. (It is slightly longer but we have other uses for it.) Assume that u solves $u' + Au = 0$ and consider the equation satisfied by the pair (u, v) where $v = tu'$. Since $v' = tu'' + u' = -(tAu' + Au) = -(Av + Au)$, we have:

$$(u, v)' + \mathcal{A}(u, v) = 0.$$

This translates into the expected relationship

$$(2.9) \quad \mathcal{S}(t)(x, y) = (S(t)x, S(t)y - tAS(t)x)$$

between the semigroup \mathcal{S} generated by $-\mathcal{A}$ and the semigroup S generated by $-A$ (if these exist). Given a strongly continuous \mathcal{S} generated by $-\mathcal{A}$, (2.9) defines S by setting $S(t)y$ equal to the second component of $\mathcal{S}(t)(0, y)$. It is easy to check that S is analytic (since $tAS(t)$ will have to be bounded) and is generated by $-A$. Similarly if S is analytic and generated by $-A$ then \mathcal{S} is strongly continuous and generated by $-\mathcal{A}$. We verify only the strong continuity $\lim_{t \rightarrow 0+} \mathcal{S}(t)(x, y) = (x, y)$. For this it suffices to have $\lim_{t \rightarrow 0+} tAS(t)x = 0$ for all

$x \in X$. However $\lim_{t \rightarrow 0^+} tAS(t)x = \lim_{t \rightarrow 0^+} tS(t)Ax = 0$ for $x \in D(A)$ and $tAS(t)$ is uniformly bounded for $0 < t < 1$, whence the result. ■

REMARK. It is obvious that $t\mathcal{A}\mathcal{S}(t)$ is bounded for $0 < t < 1$ if S is analytic. Thus $-\mathcal{A}$ generates a strongly continuous semigroup only if it generates an analytic semigroup.

Since the proof of Theorem 2 was simple, we cannot expect this result to substantially reduce the task of showing particular generators generate analytic semigroups. However, Theorem 2 does suggest the sufficient condition below, which is employed in the next section.

Let $\mathcal{F} : X \times X \rightarrow \mathbf{R}$ be continuously differentiable. We write $\mathcal{F}(x, y)$ for $(x, y) \in X \times X$ and $\mathcal{F}_x(x, y)$, $\mathcal{F}_y(x, y)$ for the gradient maps $\mathcal{F}_x, \mathcal{F}_y : X \times X \rightarrow X^*$ obtained by differentiating $x \rightarrow \mathcal{F}(x, y)$ and $y \rightarrow \mathcal{F}(x, y)$ respectively. The notation $\langle x^*, x \rangle$ means the value of $x^* \in X^*$ at $x \in X$.

PROPOSITION 3. Let $-\mathcal{A}$ be the infinitesimal generator of a strongly continuous semigroup S on X . Let $\mathcal{F} : X \times X \rightarrow \mathbf{R}$ be continuously differentiable and satisfy

$$(2.10) \quad \begin{cases} \text{There are positive constants } a_0, a_1, \alpha \text{ such that} \\ a_0 \|(x, y)\|^\alpha \leq \mathcal{F}(x, y) \leq a_1 \|(x, y)\|^\alpha \quad \text{for } (x, y) \in X \times X. \end{cases}$$

If

$$(2.11) \quad \langle \mathcal{F}_x(x, y) + \mathcal{F}_y(x, y), Ax \rangle + \langle \mathcal{F}_y(x, y), Ay \rangle \geq 0$$

for $(x, y) \in D(A) \times D(A)$,

then S is analytic.

REMARK. We have not specified a norm on $X \times X$, but (2.10) is invariant under a change of norm so this is not important.

PROOF. \mathcal{S} , as given by (2.9), is a semigroup on $D(A^2) \times D(A)$ and $\mathcal{S}(t)(x, y)$ is continuously differentiable in t for $(x, y) \in D(A^2) \times D(A)$. Moreover, by the chain rule and (2.11),

$$\frac{d}{dt} \mathcal{F}(\mathcal{S}(t)(x, y)) \Big|_{t=0} = -\langle \mathcal{F}_x(x, y) + \mathcal{F}_y(x, y), Ax \rangle - \langle \mathcal{F}_y(x, y), Ay \rangle \leq 0$$

for $(x, y) \in D(A^2) \times D(A)$. By the semigroup property of \mathcal{S} ,

$$\frac{d}{dt} \mathcal{F}(\mathcal{S}(t)(x, y)) \leq 0 \quad \text{for } t \geq 0$$

and by (2.10)

$$(2.12) \quad a_0^{1/\alpha} \|\mathcal{S}(t)(x, y)\| \leq \mathcal{F}(\mathcal{S}(t)(x, y))^{1/\alpha} \leq \mathcal{F}(x, y)^{1/\alpha} \leq a_1^{1/\alpha} \|(x, y)\|$$

and $\mathcal{S}(t)$ is bounded (uniformly in t) on $D(A^2) \times D(A)$. Thus $\|tAS(t)\|$ is bounded for $t > 0$ and the proof is complete. ■

REMARK. The technical conditions of Proposition 3 can be perturbed in many ways to obtain similar results. Our main interest here is in the general idea that it suffices to exhibit a functional (satisfying some conditions) which decreases along trajectories of \mathcal{S} in order to know that S is analytic.

§3. An application

It is known that strongly elliptic operators generate analytic semigroups in L^p , $1 \leq p < \infty$ and in spaces of continuous functions. The case of elliptic operators in $L^p(\Omega)$, $1 < p < \infty$ for Ω bounded with general boundary conditions is due to S. Agmon [1] (see also [6], [8]). In the continuous function spaces with Ω bounded or unbounded and with Dirichlet boundary conditions the results are due to H. Stewart [13]. These results were obtained by verifying the complex resolvent estimates required using the method of S. Agmon [1]. T. Kato (personal communication) and, independently, F. Massey have observed that analyticity in L^1 can be obtained via duality with the continuous function case.

In so far as we are aware, higher order strongly elliptic operators are known to generate strongly continuous semigroups in L^p , $1 < p < \infty$, $p \neq 2$ only by the complex resolvent estimates which show that they generate analytic semigroups.

For second order elliptic operators one can give a direct argument based on multiplication by appropriate functions, that shows that they satisfy the complex resolvent estimates required for the operator to generate an analytic semigroup in L^p , $1 < p < \infty$, see [12]. In this case, however, one does not need these results to show that second order elliptic operators generate strongly continuous semigroups, since this can be shown in a simpler way (see e.g. [5]). One is then naturally interested in a direct proof of the analyticity of those semigroups which does not depend on the complex resolvent estimates. This is our next topic.

Let Ω be a bounded domain in \mathbf{R}^N with a smooth boundary $\partial\Omega$. Let A be a symmetric second order differential operator formally given by

$$(3.1) \quad A = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \cdot \right)$$

We assume that the coefficients $a_{ij}(x)$ are real, continuously differentiable in $\bar{\Omega}$ and that A is uniformly elliptic, i.e. there are constants $C_0, C_1 > 0$ such that

$$(3.2) \quad C_1 |\xi|^2 \cong \sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j \cong C_0 |\xi|^2 \quad \text{for } x \in \Omega, \quad \xi \in \mathbf{R}^N.$$

We consider A as an operator in $L^p(\Omega)$, $1 \leq p < \infty$, with domain

$$D(A) = \{u : u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\}$$

where $W^{m,p}(\Omega)$ is the usual Sobolev space of distributions which have m derivatives in $L^p(\Omega)$ and $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

It is not difficult to see that $-A$ generates a strongly continuous semigroup of contraction $S_p(t)$ in $L^p(\Omega)$ for $1 \leq p < \infty$ (see e.g. [5]). We will show using Proposition 3 that $S_p(t)$ is an analytic semigroup for $2 \leq p < \infty$ (and then, for example, by duality also for $1 < p < 2$).

Let $F: \mathbf{R}^2 \rightarrow \mathbf{R}$ be such that

$$(3.3) \quad (u, v) \rightarrow \mathcal{F}(u, v) = \int_{\Omega} F(u(x), v(x)) dx$$

is continuously differentiable on $L^p(\Omega) \times L^p(\Omega)$. Then for $u, v \in D(A)$ we have

$$\begin{aligned} \langle \mathcal{F}_u(u, v) + \mathcal{F}_v(u, v), Au \rangle + \langle \mathcal{F}_v(u, v), Av \rangle \\ = \int_{\Omega} [(F_u(u, v) + F_v(u, v))Au + F_v(u, v)Av] dx \end{aligned}$$

where the dual of $L^p(\Omega)$ is taken as $L^{p/(p-1)}(\Omega)$ and F_u, F_v denote the derivatives of F with respect to its first and second arguments respectively. Since $u, v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ we can integrate the last equality by parts (provided that F has enough properties to justify this) and use (3.2) to obtain

$$\begin{aligned} \langle \mathcal{F}_u(u, v) + \mathcal{F}_v(u, v), Au \rangle + \langle \mathcal{F}_v(u, v), Av \rangle &= \\ &= \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x) \left[(F_{uu}(u, v) + F_{vu}(u, v)) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right. \\ &\quad \left. + (2F_{uv}(u, v) + F_{vv}(u, v)) \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_j} + F_{vv}(u, v) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \right] dx \\ &\geq \int_{\Omega} \left[C_0 \left\{ (F_{uu} + F_{vu}) \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i} \right)^2 + F_{vv} \sum_{i=1}^N \left(\frac{\partial v}{\partial x_i} \right)^2 \right\} \right. \\ &\quad \left. - C_1 |2F_{uv} + F_{vv}| \left(\sum_{i=1}^N \left(\frac{\partial u}{\partial x_i} \right)^2 \sum_{i=1}^N \left(\frac{\partial v}{\partial x_i} \right)^2 \right)^{1/2} \right] dx \end{aligned} \quad (3.4)$$

provided that $F_{uu} + F_{vu} \geq 0$ and $F_{vv} \geq 0$.

The integrand of the right hand side of (3.4) will be nonnegative for all $u, v \in D(A)$ if the following conditions are satisfied:

$$(3.5) \quad \begin{aligned} & \text{(i)} \quad F_{vv} \geq 0, \\ & \text{(ii)} \quad F_{uu} + F_{vv} \geq 0, \\ & \text{(iii)} \quad 4R^2 F_{uv}^2 + 4(R^2 - 1)F_{uv}F_{vv} + R^2 F_{vv}^2 \leq 4F_{uu}F_{vv}, \quad R = \frac{C_1}{C_0}. \end{aligned}$$

To complete our proof that $S_p(t)$ is analytic we have to find a function F such that \mathcal{F} in (3.3) is continuously differentiable, the above calculations are valid, (3.5) holds and (2.10) holds with $X = L^p(\Omega)$.

To this end, assume that F is positive homogeneous of degree $p \geq 2$ and write (3.5) for $F_\lambda(u, v) = F(u, \lambda v)$ where $\lambda > 0$. The conditions (3.5)(i), (ii) are invariant under this change of variables while (3.5)(iii) becomes

$$4\lambda^2 R^2 F_{uv}^2 + 4(R^2 - 1)\lambda^3 F_{uv}F_{vv} + R^2 \lambda^4 F_{vv}^2 \leq 4\lambda^2 F_{uu}F_{vv}.$$

Dividing by λ^2 , letting $\lambda \rightarrow 0$ and recalling the homogeneity, we see that it suffices to find F such that

$$(3.6) \quad 4R^2 F_{uv}^2 < 4F_{uu}F_{vv} \quad \text{for } u^2 + v^2 = 1$$

and then (3.5)(iii) will hold for F_λ if $\lambda > 0$ is sufficiently small. We choose

$$(3.7) \quad F(u, v) = (\alpha |u|^p + \alpha |v|^p + |u + v|^p), \quad \alpha > 0.$$

For $p \geq 2$, F and \mathcal{F} have the regularity required above while α can be chosen so large that (3.6) holds. Thus S_p is analytic for $p \geq 2$.

REMARKS. (1) For the special case of the Laplacian, one can compute a constant C in (4) explicitly. Using the function

$$F(u, v) = p^{-1}(u^2 + \lambda v^2)^{p/2}$$

with $\lambda \leq \min(4(p-1), 4(p-1)^{-1})$ one can check that the conditions (3.5) are satisfied and one finds

$$(3.8) \quad \|\Delta S_p(t)\| \leq \left(\frac{1}{2\sqrt{p-1}} \right) \frac{1}{t} \quad \text{for } 1 < p \leq 2$$

and

$$(3.9) \quad \|\Delta S_p(t)\| \leq \left(\frac{\sqrt{p-1}}{2} \right) \frac{1}{t} \quad \text{for } 2 \leq p < \infty.$$

(2) The above proof obviously generalizes in many ways. For example, one can handle various boundary conditions and lower order terms. However, more interesting questions are: Can something similar be done for $p = 1$? For higher order operators? Or in the abstract setting of [11, theor. X, 43]? This last reference is to an abstract theorem (which applies to our example) which is proved roughly by interpolating between strongly continuous in L^1 and analytic in L^2 by complex methods. Our proof is similar to that of [2] for $p = 2$.

Finally we observe that the \mathcal{F} we constructed in this case is essentially a norm ($\mathcal{F}^{1/p}$ is a norm), but we did not have to verify this. Moreover, it is possible to give reasonable \mathcal{F} 's which do not correspond to norms.

Appendix. A nonlinear related result

Let H be a real Hilbert space and let A be a maximal monotone operator in H . For the definition and elementary properties of maximal monotone operators used in this section see e.g. [3]. It is well known that a maximal monotone operator A in H generates a semigroup $S(t)$ of nonexpansive mappings on $\overline{D(A)}$. We say that $S(t)$ has the regularizing property if $S(t)\overline{D(A)} \subset D(A)$ for $t > 0$. (This corresponds to $S(t)$ being a C^∞ semigroup in the linear case.) For maximal monotone operators $J_\lambda = (I + \lambda A)^{-1}$ is a nonexpansive mapping of H into itself. In this appendix we will prove that if $A = \partial\varphi$ is the subdifferential of a convex lower semicontinuous function φ then

$$(A.1) \quad \|A^0 J_\lambda^n x\| \leq \frac{C}{n\lambda} \quad \text{for } \lambda > 0, \quad n\lambda \leq 1 \quad \text{and} \quad n = 1, 2, \dots$$

where $A^0 x$ is, as usual, the element of minimum norm in the set Ax . Thus if $A = \partial\varphi$ it satisfies the nonlinear analogue of the condition of Theorem 1, and we therefore expect that it generates a semigroup with the regularizing property. That this is indeed the case follows easily from (A.1) together with the nonlinear analogue of (3). (A.1) is a straightforward consequence of the following more general proposition which is a modification of a result of [4].

PROPOSITION. *Let φ be a convex lower semicontinuous function on H and let $A = \partial\varphi$ be the subdifferential of φ . For every $x_0 \in H$ and $\{\lambda_n\} \subset \mathbf{R}^+$ let*

$$(A.2) \quad x_n = J_{\lambda_n} x_{n-1}, \quad n = 1, 2, \dots$$

Then for every $[u, w] \in A$

$$(A.3) \quad \|A^0 x_n\|^2 \leq \left(\sum_{k=1}^n \lambda_k \right)^{-2} \|x_0 - u\|^2 - 2 \left(\sum_{k=1}^n \lambda_k \right)^{-1} (w, x_n - u).$$

PROOF. (A.2) can be also written as

$$(A.2)' \quad x_{n-1} = x_n + \lambda_n y_n, \quad y_n \in Ax_n.$$

We denote by Ax_n the element $y_n \in Ax_n$ which is defined by (A.2)'. From the monotonicity of A it follows that

$$0 \leq (Ax_{k-1} - Ax_k, x_{k-1} - x_k) = \lambda_k (Ax_{k-1} - Ax_k, Ax_k)$$

and therefore

$$(A.4) \quad \|Ax_k\| \leq \|Ax_{k-1}\| \quad \text{for } k = 2, 3, \dots$$

Denoting

$$\sigma_0 = 0, \quad \sigma_n = \sum_{k=1}^n \lambda_k$$

we have for every $u \in D(A)$

$$\begin{aligned} \|x_{k-1} - u\|^2 &= \|x_k - u\|^2 + 2\lambda_k (Ax_k, x_k - u) + \lambda_k^2 \|Ax_k\|^2 \\ &\geq \|x_k - u\|^2 + 2\lambda_k (\varphi(x_k) - \varphi(u)) + \lambda_k^2 \|Ax_k\|^2 \\ (A.5) \quad &\geq \|x_k - u\|^2 + 2\lambda_k \sigma_{k-1} \|Ax_k\|^2 + 2\sigma_k \varphi(x_k) \\ &\quad - 2\sigma_{k-1} \varphi(x_{k-1}) - 2\lambda_k \varphi(u) + \lambda_k^2 \|Ax_k\|^2 \end{aligned}$$

where the first equality follows from (A.2)', the first inequality follows from the definition of the subdifferential and the second inequality follows from

$$\sigma_{k-1} \varphi(x_{k-1}) - \sigma_k \varphi(x_k) + \lambda_k \varphi(x_k) = \sigma_{k-1} (\varphi(x_{k-1}) - \varphi(x_k)) \geq \sigma_{k-1} \lambda_k \|Ax_k\|^2.$$

Summing (A.5) from $k = 1$ to $k = n$ and using (A.4) we find

$$\begin{aligned} \|x_0 - u\|^2 &\geq \|x_n - u\|^2 + \sum_{k=1}^n (2\lambda_k \sigma_{k-1} + \lambda_k^2) \|Ax_k\|^2 + 2\sigma_n (\varphi(x_n) - \varphi(u)) \\ &\geq \|x_n - u\|^2 + \sigma_n^2 \|Ax_n\|^2 + 2\sigma_n (Au, x_n - u) \end{aligned}$$

and (A.3) follows at once. ■

To deduce (A.1) from (A.3) choose $u = x_n$ and $\lambda_k = \lambda$ for $k = 1, 2, \dots$ in (A.3). Then

$$(A.6) \quad \|A^0 x_n\| \leq \|Ax_n\| \leq \frac{\|x_0 - x_n\|}{n\lambda}.$$

If $n\lambda \leq 1$ then $\|x_0 - x_n\| \leq C$. Indeed, for every $u \in D(A)$ we have

$$\|x_0 - x_n\| \leq 2\|x_0 - u\| + \|u - J_{\lambda}^n u\| \leq 2\|x_0 - u\| + n\lambda \|A^0 u\|$$

and (A.1) follows.

REMARK. If we assume that $F = A^{-1}0 \neq \emptyset$ and denote by P the projection on F we obtain by choosing $u = Px_0$ in (A.3)

$$(A.7) \quad \|Ax_n\| \leq \left(\sum_{k=1}^n \lambda_k \right)^{-1} \text{dist}(x_0, F)$$

which is slightly better (having a better constant) than the result obtained in [4].

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